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KILLING-YANO SUPERSYMMETRY IN STRING THEORY

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ABSTRACT

The presence of Killing-Yano tensors implies the existence of non-standard supersymmetries in point particle theories on curved backgrounds. In a string theoretical context these are symmetries of the modes describing the particle-like behavior of the string. In the presence of isometries we show that, in addition to these, one can also define a new type of non-standard supersymmetry among a mixture of particle and winding modes. The interplay with T-duality is also examined and illustrated by explicit examples.

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1 Introduction

Extended $N = 2$ and $N = 4$ supersymmetry (in addition to the usual $N = 1$) in 2-dim σ -models has been investigated in detail [1] leading to powerful structures that have applications in many branches of modern theoretical physics from supergravity and string theory to gravitational instantons and moduli problems in monopole physics.

On the other hand it was pointed out in [2] that in particle models with $N = 1$ world-line supersymmetry (so-called spinning particle models), the conditions for the existence of additional supersymmetries are different and in fact less stringent than the corresponding ones on the world-sheet. Roughly speaking, they require instead of a covariantly constant complex structure to exist on the target space only a so called Killing-Yano tensor, satisfying $D_\mu F_{\nu\rho} + D_\nu F_{\mu\rho} = 0$ [3]. This new type of extended supersymmetry does not obey the standard algebra and has some unusual as well as appealing features. It can exist in odd dimensional target spaces, as opposed to the standard $N = 2$ extended supersymmetry which requires the dimension to be even, or to $N = 4$ which restricts it to be a multiple of four. In addition in 4-dim the signature can be Lorentzian as opposed to the standard extended supersymmetry which only admits solutions in Euclidean or Kleinian signature target spaces [2]. Even more intriguing is the fact that the entire family of Kerr-Newman black holes has this supersymmetry [2], whereas only the extremally charged members have supersymmetry of the usual type in the context of supergravity theory.

Since the 2-dim σ -model reduces to a particle model as far as the center of mass motion of the string is concerned, such Killing-Yano tensors define an approximate extended supersymmetry of the σ -model among only the particle-like string modes. Intuitively, under T-duality, any symmetry among the particle-like string modes should transform into a symmetry of winding modes. In this note we investigate the appearance of these extended supersymmetries in the winding sector and their relation to Target space duality (T-duality) [4]. The latter interpolates between effective field theories corresponding to different backgrounds and its interplay with ordinary supersymmetry has been fruitful in revealing string phenomena that resolve paradoxes with field theoretical origin [5].

The organisation of this paper is as follows: In section 2 we set up our general framework in the Hamiltonian formalism, for $N = 1$ as well as extended supersymmetry, for any 2-dim σ -model (technical details can be found in the appendix). Section 3 contains our main results concerning the existence of non-standard extended supersymmetry among particle-like string modes and among winding-particle-like modes. The behavior of these supersymmetries under T-duality is also examined in general and by working out the details in the 3-dim flat space and the 4-dim Taub-NUT metric and their T-duals. We also point out, with an example, that Killing-Yano type supersymmetry in a string context might be important in relation to S-duality.

2 The 2D σ -model

The action, in the component formalism, of a 2-dim σ -model with $N = 1$ world-sheet supersymmetry is [1]:

$$S(X, \Psi_+, \Psi_-) = \frac{1}{2} \int Q_{\mu\nu}^+ \partial_+ X^\mu \partial_- X^\nu + i G_{\mu\nu} \Psi_+^\mu \left(\partial_- \Psi_+^\nu + (\Omega^+)^{\nu}_{\lambda\rho} \partial_- X^\lambda \Psi_+^\rho \right) \\ + i G_{\mu\nu} \Psi_-^\mu \left(\partial_+ \Psi_-^\nu + (\Omega^-)^{\nu}_{\lambda\rho} \partial_+ X^\lambda \Psi_-^\rho \right) + \frac{1}{2} R_{\mu\nu\rho\lambda}^- \Psi_+^\mu \Psi_+^\nu \Psi_-^\rho \Psi_-^\lambda, \quad (1)$$

where $G_{\mu\nu}$ and $B_{\mu\nu}$ are the metric and the antisymmetric tensor and $Q_{\mu\nu}^+ \equiv G_{\mu\nu} + B_{\mu\nu}$; for later use we also define $Q_{\mu\nu}^- \equiv G_{\mu\nu} - B_{\mu\nu}$. The generalized connections are: $(\Omega^\pm)_{\mu\nu}^\rho = \Gamma_{\mu\nu}^\rho \pm \frac{1}{2} H_{\mu\nu}^\rho$, where¹ $H_{\mu\nu\rho} \equiv 3\partial_{[\rho} B_{\mu\nu]}$ is the torsion. The corresponding curvature tensors are $R_{\mu\nu\rho}^\pm{}^\lambda = 2\partial_{[\nu} (\Omega^\pm)_{\mu]}^\lambda{}_\rho + 2(\Omega^\mp)_{\sigma[\nu}^\lambda (\Omega^\pm)_{\mu]\rho}^\sigma = R_{\rho\lambda\mu\nu}^\mp$. Integration is over $\sigma^\pm = \frac{1}{2}(\sigma \pm \tau)$, where σ and τ are the natural spatial and time coordinates on the world-sheet. Since our considerations will be mainly classical, the dilaton term has been omitted.

Our framework is based on the Hamiltonian formalism. The only complication in the transition from the Lagrangian to the Hamiltonian formalism is that we have to deal with the second class constraints associated with the fermions. A straightforward application of Dirac's treatment of constraint systems yields for the Hamiltonian²

$$H = \frac{1}{2} G^{\mu\nu} \Pi_\mu \Pi_\nu + \frac{1}{2} G_{\mu\nu} X'^\mu X'^\nu + \frac{i}{2} G_{\mu\nu} \left(\Psi_+^\mu \Psi_+^\nu - \Psi_-^\mu \Psi_-^\nu \right) \\ + \frac{i}{2} (\Omega^+)_{\mu;\lambda\rho} \Psi_+^\mu \Psi_+^\rho X'^\lambda - \frac{i}{2} (\Omega^-)_{\mu;\lambda\rho} \Psi_-^\mu \Psi_-^\rho X'^\lambda - \frac{1}{4} R_{\mu\nu\rho\lambda}^- \Psi_+^\mu \Psi_+^\nu \Psi_-^\rho \Psi_-^\lambda, \quad (2)$$

where $(\Omega^\pm)_{\mu;\nu\lambda} \equiv G_{\mu\rho} (\Omega^\pm)_{\nu\lambda}^\rho$ and $\Pi_\mu \equiv G_{\mu\nu} \dot{X}^\nu$. The conjugate momentum to X^μ is

$$P_\mu = \Pi_\mu - B_{\mu\nu} X'^\nu + \frac{i}{2} \Omega_{\nu;\mu\rho}^+ \Psi_+^\nu \Psi_+^\rho + \frac{i}{2} \Omega_{\nu;\mu\rho}^- \Psi_-^\nu \Psi_-^\rho, \quad (3)$$

and the non-vanishing basic Poisson (actually Dirac) brackets are found to be

$$\{X^\mu(\sigma), P_\nu(\sigma')\} = \delta^\mu_\nu \delta(\sigma, \sigma'), \quad \{\Psi_\pm^\mu(\sigma), \Psi_\pm^\nu(\sigma')\} = -i G^{\mu\nu} \delta(\sigma, \sigma'), \\ \{P_\mu(\sigma), \Psi_\pm^\nu(\sigma')\} = \frac{1}{2} G_{\rho\lambda,\mu} G^{\lambda\nu} \Psi_\pm^\rho \delta(\sigma, \sigma'), \quad (4) \\ \{P_\mu(\sigma), P_\nu(\sigma')\} = -\frac{i}{4} G^{\rho\lambda} G_{\rho\alpha,\mu} G_{\lambda\beta,\nu} (\Psi_+^\alpha \Psi_+^\beta + \Psi_-^\alpha \Psi_-^\beta) \delta(\sigma, \sigma').$$

It is worth noticing that the bracket between two momenta P_μ and P_ν is non-vanishing. In various manipulations that will follow we found convenient to use the combinations Π_μ and $\Pi_{\pm\mu} \equiv G_{\mu\nu} \partial_\pm X^\nu$ instead of P_μ . Some additional Poisson brackets that proved useful are:

$$\{\Pi_\mu(\sigma), \Psi_\pm^\nu(\sigma')\} = \{\Pi_{\pm\mu}(\sigma), \Psi_\pm^\nu(\sigma')\} = (\Omega^\pm)_{\mu\rho}^\nu \Psi_\pm^\rho \delta(\sigma, \sigma'), \\ \{\Pi_\mu(\sigma), \Pi_\nu(\sigma')\} = \frac{i}{2} \left(R_{\mu\nu\rho\lambda}^+ \Psi_+^\rho \Psi_+^\lambda + R_{\mu\nu\rho\lambda}^- \Psi_-^\rho \Psi_-^\lambda + 2i H_{\mu\nu\lambda} X'^\lambda \right) \delta(\sigma, \sigma'). \quad (5)$$

¹Round brackets (...) denote complete symmetrisation over all indices with total weight equal to one. Similarly, square brackets [...] denote complete antisymmetrisation.

²In the rest of the paper integration over the spatial variable σ , wherever necessary, is understood. We will also denote derivatives with respect to τ and σ by a dot and a prime respectively.

The action (1) is invariant under the standard $N = 1$ supersymmetry transformations in the two chiral sectors

$$\begin{aligned}\delta X^\mu &= i\epsilon_- \{Q_+, X^\mu\} + i\epsilon_+ \{Q_-, X^\mu\} \\ \delta \Psi_\pm^\mu &= i\epsilon_\mp \{Q_\pm, \Psi_\pm^\mu\} + i\epsilon_\pm \{Q_\mp, \Psi_\pm^\mu\},\end{aligned}\tag{6}$$

where ϵ_\pm are constant anticommuting parameters and the supercharges are given by

$$Q_\pm = -\Pi_{\pm\mu} \Psi_\pm^\mu \pm \frac{i}{6} H_{\mu\nu\rho} \Psi_\pm^\mu \Psi_\pm^\nu \Psi_\pm^\rho.\tag{7}$$

The corresponding supersymmetry algebra is:

$$\{Q_\pm, Q_\pm\} = -2iH_\pm, \quad \{Q_+, Q_-\} = 0, \quad \{Q_\pm, H_\pm\} = \{Q_\pm, H_\mp\} = 0,\tag{8}$$

where

$$\begin{aligned}H_\pm &= \frac{1}{2} G^{\mu\nu} \Pi_\mu \Pi_\nu + \frac{1}{2} G_{\mu\nu} X'^\mu X'^\nu \pm X'^\mu \Pi_\mu \pm i G_{\mu\nu} \Psi_\pm^\mu \Psi_\pm^{\nu'} \\ &\quad \pm i(\Omega^\pm)_{\mu;\lambda\rho} \Psi_\pm^\mu \Psi_\pm^\rho X'^\lambda - \frac{1}{4} R_{\mu\nu\rho\lambda}^- \Psi_\pm^\mu \Psi_\pm^\nu \Psi_\pm^\rho \Psi_\pm^\lambda.\end{aligned}\tag{9}$$

As we have already mentioned, the Hamiltonian $H = \frac{1}{2}(H_+ + H_-)$ that generates time-translations on the world-sheet is given by (2), whereas the generator of the space-translations is $\frac{1}{2}(H_+ - H_-)$.

The search for new supersymmetries starts by making the Ansatz that, in addition to (7), extra supercharges of the form

$$Q_\pm^F = -\Pi_{\pm\mu} (F^\pm)^\mu{}_\nu \Psi_\pm^\nu \pm \frac{i}{6} C_{\mu\nu\rho}^\pm \Psi_\pm^\mu \Psi_\pm^\nu \Psi_\pm^\rho,\tag{10}$$

exist in each chiral sector separately. The conditions on $F_{\mu\nu}^\pm$ are well known [1]; the necessary algebra in our conventions can be found in appendix A. One finds that the target space tensor $(F^\pm)^\mu{}_\nu$ is an (almost) complex (hermitian) structure, which satisfies the antisymmetry condition

$$F_{\mu\nu}^\pm + F_{\nu\mu}^\pm = 0,\tag{11}$$

is covariantly constant with respect to the generalized connection

$$D_\mu^\pm (F^\pm)^\rho{}_\nu = 0,\tag{12}$$

and squares to $-\mathbf{1}$,

$$(F^\pm)^\mu{}_\lambda (F^\pm)^\lambda{}_\nu = -\delta^\mu{}_\nu.\tag{13}$$

In addition $C_{\mu\nu\rho}^\pm$ is completely determined as³

$$C_{\mu\nu\rho}^\pm = 3H_{\lambda[\mu\nu} (F^\pm)^\lambda{}_{\rho]}.\tag{14}$$

³ We have also used that

$$R_{\mu\nu\rho\alpha}^\pm (F^\pm)^\alpha{}_\lambda = R_{\mu\nu\lambda\alpha}^\pm (F^\pm)^\alpha{}_\rho, \quad \partial_{[\mu} C_{\nu\rho\lambda]}^\pm = 0.$$

The first equation is the integrability condition for (12) whereas the second equation follows by explicitly writing out the identity $d^2 F^\pm = 0$.

Then the original $N = 1$ world-sheet supersymmetry is promoted to an $N = 2$ one. In cases where there exist three independent complex structures in each sector the supersymmetry is actually an $N = 4$ one. The above requirements put severe restrictions on backgrounds that admit a solution, as it has been extensively discussed in [1]; for instance, in the absence of torsion $N = 2$ ($N = 4$) extended supersymmetry requires Kähler (hyper-Kähler) manifolds.

3 Non-standard extended supersymmetry and T-duality

We would like to investigate the possibility of having additional extended supersymmetries, not of the standard type, by modifying some of the equations (11)–(13). One possibility, is to allow the complex structures to depend non-locally on the target space variables. This arose naturally [5] in investigations on the interplay between Abelian [6, 5, 7, 8] and non-Abelian [9] T-duality and world-sheet supersymmetry and has its counterpart in conformal field theory since non-local complex structures are directly related [5, 8, 9] to coset parafermions. Such complex structures are not covariantly constant [5, 7] and equation (12) is modified by a term given in general in [8].

Here we pursue an alternative line of investigation. We still insist on the existence of local target space tensors $F^\mu{}_\nu$, but we relax the condition that the new supersymmetry corresponding to it is a symmetry of the entire string spectrum. We will essentially restrict to the particle-like part of the spectrum as well as to the so called winding-particle-like part.

3.1 Supersymmetry for the particle-like modes

Restricting to the modes that describe the particle behavior of the string implies that we neglect all oscillatory modes for the bosonic as well as for the fermionic degrees of freedom. Hence, we consider the limit⁴

$$X'^\mu = 0, \quad \Psi'^\mu_- = 0, \quad \Psi^\mu_+ = 0. \quad (15)$$

which will call for brevity the particle limit. We refer the reader to the appendix for the computational details and present the results here. We find the generalisation to nonzero torsion of the well known result [2] that the conditions for an extra supersymmetry are (11) and a modified version of (12) given by

$$D_\mu^- F_{\rho\nu}^- + D_\nu^- F_{\rho\mu}^- = 0, \quad (16)$$

(see also [10]). The corresponding supercharge Q_-^F is given by (7), with

$$C_{\mu\nu\rho}^- = -2D_{[\mu}^- F_{\nu\rho]}^- + 3H_{\lambda[\mu\nu}(F^-)^\lambda{}_\rho], \quad (17)$$

⁴ Without loss of generality we focus attention to one chiral sector only. Hence, half of the fermionic degrees of freedom in (15) and later in (22) are set to zero.

and it should obey ⁵

$$\partial_{[\mu} C_{\nu\rho\lambda]}^- = 0 . \quad (18)$$

This condition was also encountered in the case of extended supersymmetry in the full 2–dim model (see footnote 3), but unlike this case here it actually leads to a restriction for the string torsion which, after using (17), reads

$$\partial_{[\mu} \left(H_{|\alpha|\nu\rho} (F^-)^{\alpha}{}_{\lambda]} \right) = 0 . \quad (19)$$

In the case of zero torsion (16), (17) reduce to the conditions for the existence of a Yano tensor. Explicit examples for such tensors can be found in [2, 11]. Notice that condition (13) is not a requirement, since the supercharge corresponding to the new non–standard supersymmetry, doesn’t have to square to the Hamiltonian [2] (cf. (8)).

In our string inspired discussion, we have assumed so far that the torsion was integrable, i.e. $H = dB$. Since spinning particle models are interesting in their own right it is important to know the analogs of the conditions (18), (19) for existence of extended Yano–type world–line supersymmetry in general models, where the torsion is not integrable. Sparing the reader from all computational details, it turns out that (18) is modified to

$$\partial_{[\mu} C_{\nu\rho\lambda]}^- + \frac{1}{2} F_{\alpha[\mu\nu\rho} (F^-)^{\alpha}{}_{\lambda]} = 0 , \quad F_{\mu\nu\rho\lambda} \equiv 4 \partial_{[\mu} H_{\nu\rho\lambda]} . \quad (20)$$

Consequently the condition on the torsion (19) is generalised to

$$\partial_{[\mu} \left(H_{|\alpha|\nu\rho} (F^-)^{\alpha}{}_{\lambda]} \right) + \frac{1}{2} F_{\alpha[\mu\nu\rho} (F^-)^{\alpha}{}_{\lambda]} = 0 . \quad (21)$$

The above generalisation is trivial in 4–dim target spaces. The reason is that in 4–dim the tensor $F_{\mu\nu\rho\lambda}$ has only one independent component and as a consequence the extra terms in (20), (21) (as compared to (18), (19)) are proportional to the trace of the Yano tensor $(F^-)^{\mu}{}_{\mu} = 0$. Finally, it is interesting that adding abelian gauge fields doesn’t affect the above conditions at all. It restricts however the gauge field strength similarly to what was found in the last reference in [11].

In the rest of the paper we continue our discussion with integrable torsion.

3.2 Supersymmetry for the winding–particle–modes

In the presence of isometries (we will, for simplicity, only consider the case of one Abelian isometry) one expects on the basis of T–duality that a similar non–standard extra supersymmetry should appear in the modes dual to the particle modes. We will work in an adapted coordinate system, with Killing vector field $\partial/\partial X^0$, where the background fields are X^0 –independent. Then T–duality boils down to

⁵We also have used the integrability condition for (16) which, after using (18), reads

$$D_{\mu}^{-} C_{\alpha\beta\gamma}^{-} = 3 R_{\mu\rho[\alpha\beta}^{-} (F^{-})^{\rho}{}_{\gamma]} .$$

a canonical transformation [12] that interchanges $X'^0 \leftrightarrow P_0$ (notice, not $\Pi_0!$). Therefore we may try to consider instead of the particle limit (15) what we will call winding-particle limit given by

$$P_0 = 0 , \quad X'^i = 0 , \quad \Psi'^\mu_- = 0 , \quad \Psi^\mu_+ = 0 . \quad (22)$$

Notice that, because of the Killing symmetry this is consistent with the Poisson brackets (4) as it should be. Again referring the reader to the appendix for the computational details we have found that the necessary conditions for a non-standard extended supersymmetry in the winding-particle-modes are equation (11) and

$$\begin{aligned} D^-_\nu F^-_\mu ({}^i G^j)^\nu &= 0 , \\ D^-_{[k} F^-_{0]j} G^{ki} &= 0 , \\ D^-_0 F^-_{0i} &= 0 , \end{aligned} \quad (23)$$

whereas the expression for $C^-_{\mu\nu\rho}$ can be obtained from

$$\begin{aligned} C^-_{\mu\nu}{}^i &= -2D^-_{[\mu} F^-_{\nu]}{}^i + 3H_{\lambda[\mu\nu}(F^-)^\lambda{}_\rho G^{\rho i} \\ C^-_{\mu\nu 0} &= -2D^-_{[\mu} F^-_{\nu]0} - D^-_0 F^-_{\mu\nu} + 3H_{\lambda[\mu\nu}(F^-)^\lambda{}_0] . \end{aligned} \quad (24)$$

Equations (23), (24) are the conditions for extra non-standard supersymmetry in the winding sector.

It should be possible to obtain the conditions (23), (24) by simply dualizing (16) directly. For notational purposes we will use tilded symbols when we refer to quantities of the dual to (1) σ -model. Let's recall the generic way to derive the transformations of the fields, by first noticing that Buscher's duality rules for the metric can be cast into the form

$$\tilde{G}_{\mu\nu} = (A_\pm)^\alpha{}_\mu (A_\pm)^\beta{}_\nu G_{\alpha\beta} \quad (25)$$

(where both signs give the same result as it should be), with

$$(A_\pm)^\mu{}_\nu = \begin{pmatrix} \pm G_{00}^{-1} & -G_{00}^{-1} Q_{j0}^\pm \\ 0 & \delta^i_j \end{pmatrix} , \quad (A_\pm^{-1})^\mu{}_\nu = \begin{pmatrix} \pm G_{00} & \pm Q_{j0}^\pm \\ 0 & \delta^i_j \end{pmatrix} . \quad (26)$$

(Q is defined below (1)). The antisymmetric tensor transforms as

$$\begin{aligned} \tilde{B}_{0i} &= \frac{G_{0i}}{G_{00}} , \\ \tilde{B}_{ij} &= B_{ij} + \frac{1}{G_{00}} (G_{i0} B_{j0} - G_{j0} B_{i0}) . \end{aligned} \quad (27)$$

In addition the transformation of the generalized connections are [7]:

$$(\tilde{\Omega}^\pm)^\mu{}_{\lambda\rho} = (A_\mp)^\alpha{}_\lambda (A_\pm)^\beta{}_\rho (A_\pm^{-1})^\mu{}_\nu (\Omega^\pm)^\nu{}_{\alpha\beta} + \partial_\lambda (A_\pm)^\alpha{}_\rho (A_\pm^{-1})^\mu{}_\alpha . \quad (28)$$

The target space fermions transform under duality as $\tilde{\Psi}_\pm^\mu = (A_\pm^{-1})^\mu{}_\alpha \Psi_\pm^\alpha$. This is deduced by demanding that the bracket between two fermions in (4), remains invariant under the duality transformation which for the metric acts as in (25). Finally, the world-sheet derivatives transform as [13, 7]

$$\partial_\pm \tilde{X}^\mu = (A_\pm^{-1})^\mu{}_\nu \partial_\pm X^\nu - i \partial_\rho (A_\pm^{-1})^\mu{}_\lambda \Psi_\pm^\rho \Psi_\pm^\lambda , \quad (29)$$

where the fermion bilinear arises after implementing T-duality as a manifestly $N = 1$ supersymmetric canonical transformation. This boson-fermion symphysis is a characteristic feature of duality in supersymmetric σ -models. Imposing that the supercharge Q_-^F in (10) has an identical form in the T-dual model, one finds that [14, 7]

$$\tilde{F}_{\mu\nu}^- = (A_-)^\alpha{}_\mu (A_-)^\beta{}_\nu F_{\alpha\beta}^- , \quad (30)$$

and

$$\tilde{C}_{\mu\nu\rho}^- = (A_-)^\alpha{}_\mu (A_-)^\beta{}_\nu (A_-)^\gamma{}_\rho C_{\alpha\beta\gamma}^- + 6(A_-)^\lambda{}_{[\nu} \partial_\mu (A_-^{-1})^\beta{}_{|\lambda} \tilde{F}_{\beta|\rho]}^- , \quad (31)$$

where indices enclosed by bars are not being antisymmetrised. Using (30), (28) one proves that [7]

$$\tilde{D}_\mu^- \tilde{F}_{\nu\rho}^- = (A_+)^\alpha{}_\mu (A_-)^\beta{}_\nu (A_-)^\gamma{}_\rho D_\alpha^- F_{\beta\gamma}^- . \quad (32)$$

Hence, in the case of complex structures obeying (12), the transformed tensors are complex structures as well. Our aim is to find the equation that the dual tensor \tilde{F}^- satisfies, provided that the tensor F^- satisfies the Killing-Yano-like condition (16), and similarly for the tensor $\tilde{C}_{\mu\nu\rho}^-$. Using (32) we find that

$$\tilde{D}_\mu^- \tilde{F}_{\lambda\rho}^- + S^\alpha{}_\mu S^\beta{}_\lambda \tilde{D}_\beta^- \tilde{F}_{\alpha\rho}^- = 0 , \quad (33)$$

where the matrix $S \equiv A_+^{-1} A_- = A_-^{-1} A_+$ is defined as

$$S^\mu{}_\nu = \begin{pmatrix} -1 & 2B_{j0} \\ 0 & \delta^i_j \end{pmatrix} \quad (34)$$

and satisfies $S^{-1} = S$. A more explicit form of (33) is

$$\begin{aligned} \tilde{D}_{(i}^- \tilde{F}_{j)k}^- - 2B_{0(i} \tilde{D}_{j)}^- \tilde{F}_{0k}^- &= 0 , \\ \tilde{D}_{[i}^- \tilde{F}_{0]j}^- &= 0 , \\ \tilde{D}_0^- \tilde{F}_{0i}^- &= 0 . \end{aligned} \quad (35)$$

The last two equations in (35) are in one to one correspondence with the similar equations in (23) (of course written with tildes). In order to compare the remaining ones we first have to contract the first equation in (35) by $\tilde{G}^{in} \tilde{G}^{jm}$ and then use the fact that $B_{0i} = \tilde{G}_{0i} / \tilde{G}_{00}$ and $\tilde{G}^{i0} = -\tilde{G}^{ij} \tilde{G}_{j0} / \tilde{G}_{00}$. After a little algebra they become indeed identical.

The last step is to check that the transformation of $C_{\mu\nu\rho}^-$ leads to equations (24). In the particle limit, the Killing-Yano tensor $F_{\mu\nu}^-$ and the completely antisymmetric tensor $C_{\mu\nu\rho}^-$ are related by (17). Using this, as well as (30) and (31) one obtains the transformed tensor:

$$\tilde{C}_{\mu\nu\rho}^- = -2\tilde{D}_\lambda^- \tilde{F}_{[\nu\rho}^- S^\lambda{}_{\mu]} - 6S^\lambda{}_{[\mu} (\tilde{\Omega}^-)^\kappa{}_{|\lambda|\nu} \tilde{F}_{|\kappa|\rho]}^- . \quad (36)$$

More explicitly

$$\begin{aligned} \tilde{C}_{ij0}^- &= -\frac{4}{3} \tilde{D}_{[i}^- \tilde{F}_{j]0}^- + \frac{2}{3} \tilde{D}_0^- \tilde{F}_{ij}^- + \tilde{H}^\kappa{}_{ij} \tilde{F}_{\kappa 0}^- - 4\tilde{\Gamma}^\kappa{}_{0[i} \tilde{F}_{j]\kappa}^- , \\ \tilde{C}_{ijm}^- &= \frac{4}{3\tilde{G}_{00}} \tilde{D}_0^- \tilde{F}_{m[j}^- \tilde{G}_{i]0} - \frac{2}{3} \frac{\tilde{G}_{0m}}{\tilde{G}_{00}} \tilde{D}_0^- \tilde{F}_{ij}^- + 3\tilde{H}^\kappa{}_{[im} \tilde{F}_{j]\kappa}^- - 2\tilde{D}_{[i}^- \tilde{F}_{jm]}^- . \end{aligned} \quad (37)$$

Manipulations similar to those mentioned below (35) together with the observation that covariant derivatives with respect to \tilde{D}_0 contain only the connection terms since $\tilde{F}_{\mu\nu}^-$ does not depend on \tilde{X}^0 , lead to the conclusion that $\tilde{C}_{\mu\nu\rho}^-$, as given by (37), indeed satisfies (24) for the dual variables, provided that conditions (35) hold.

3.3 Examples

3-dim flat space: The simplest background where a Yano tensor exists is 3-dim flat space. In cylindrical coordinates for the metric

$$ds^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2 , \quad (38)$$

it assumes the form

$$F = \rho^2 d\phi \wedge dz + \rho z d\rho \wedge d\phi . \quad (39)$$

Performing a duality transformation with respect to the vector field $\partial/\partial\phi$, one obtains for the dual metric and dilaton (we only mention it for completeness) the results

$$d\tilde{s}^2 = d\rho^2 + \frac{1}{\rho^2} d\tilde{\phi}^2 + dz^2 , \quad \tilde{\Phi} = \ln(\rho^2) \quad (40)$$

and for the dual Yano tensor

$$\tilde{F}^- = dz \wedge d\tilde{\phi} + \frac{z}{\rho} d\tilde{\phi} \wedge d\rho , \quad (41)$$

where \tilde{F}^- satisfies (23). The non-standard extended supersymmetries associated with the Yano tensor (39) and its dual (41) are the only extended supersymmetries that can exist for the flat 3-dim metric (38) and its dual (40); a usual complex structure satisfying (11)–(13) requires even dimensional backgrounds.

Taub–NUT: A more complicated example is based on the Taub–NUT metric. The line element can be cast into the following form

$$ds^2 = V(d\tau + \vec{\omega} \cdot d\vec{x})^2 + V^{-1} d\vec{x}^2 , \quad (42)$$

where τ is a Killing coordinate, $\vec{x} = (x, y, z)$ and

$$V^{-1} = \frac{1}{4} \left(1 + \frac{2m}{|\vec{x}|}\right) , \quad \vec{\omega} = \frac{mz}{2|\vec{x}|(x^2 + y^2)} (-y, x, 0) . \quad (43)$$

The Taub–NUT metric considered as a string solution admits $N = 4$ extended supersymmetry of the usual kind corresponding to the existence of three complex structures [15]. There also exists an additional supersymmetry among the particle-like modes. The corresponding Yano tensor is given by

$$F = 2(d\tau + \vec{\omega} \cdot d\vec{x}) \wedge \vec{n} \cdot d\vec{x} + \left(1 + \frac{|\vec{x}|}{m}\right) V^{-1} \epsilon_{ijk} n_i dx^j \wedge dx^k , \quad \vec{n} = \frac{\vec{x}}{|\vec{x}|} . \quad (44)$$

The dual to (42) with respect to the vector field $\partial/\partial\tau$ is an axionic instanton and the explicit expression for the metric, antisymmetric tensor and dilaton are:

$$\begin{aligned} d\tilde{s}^2 &= V^{-1} (d\tilde{\tau}^2 + d\vec{x}^2) , \\ \tilde{B} &= 2\omega_i d\tilde{\tau} \wedge dx^i , \quad \tilde{\Phi} = \ln(V) . \end{aligned} \quad (45)$$

This background has also standard $N = 4$ extended supersymmetry with complex structures given in [5]. Again, as we shown in our general framework, there exists a non-standard supersymmetry

in the winding-particle-like modes defined in (22) with dual Yano tensor satisfying (23) and given explicitly by

$$\tilde{F}^- = V^{-1} \left(-2d\tilde{\tau} \wedge \vec{n} \cdot d\vec{x} + \left(1 + \frac{|\vec{x}|}{m}\right) \epsilon_{ijk} n^i dx^j \wedge dx^k \right). \quad (46)$$

In the case of the Taub-NUT metric there is another Killing symmetry distinct from the one we have discussed. It is associated with the vector field generating rotations in the $x - y$ plane. T-duality with respect to this vector field breaks the manifest $N = 4$ supersymmetry into an $N = 2$, with the rest being realized non-locally [5]. As a general rule, if the complex structures or the Killing-Yano's transform non-trivially under the duality group they become non-local in the target space variables after duality [5]. In our case the Yano tensor (44) is a singlet under both isometries, thus remaining local in the dual model as well.

We have restricted our attention only to T-duality with respect to one Killing symmetry but the extension to cases with general Abelian or even non-Abelian group of isometries is straightforward. Without yielding any details, it is worth mentioning that under non-Abelian duality in the Taub-NUT metric with respect to the $SO(3)$ isometry, the Killing-Yano supersymmetry transforms into one for a collection of particle and momentum modes in the dual model, similarly to the one we have been discussing. The reason is that (44) is a singlet under $SO(3)$ rotations [15]. This is to be contrasted with the case of ordinary $N = 4$ extended supersymmetry which under the same $SO(3)$ -duality transformation breaks down to $N = 1$ with the rest being realized non-locally [9].

One would also like to know the behavior of Killing-Yano supersymmetry under S-duality. We have not investigated this question in full generality so that we will resort to an example to point out important differences with the cases of usual extended supersymmetry. Let us recall that ordinary extended supersymmetry may be destroyed under S-duality as shown with examples in [6] and its fate is not clear as is with T-duality where, as already mentioned, it just becomes non-locally realized. Let us consider 4-dim flat space with metric given by (38) plus the term $d\tau^2$ corresponding to the fourth coordinate. A series of T-S-T transformations, where T-duality is performed with respect to the rotational Killing vector field $\partial/\partial\phi$ gives again a vacuum solution to Einstein's equations in accordance with the general statement in [6] where the equivalence of this sequence of dualities and the Geroch transformation on Einstein vacuum solutions with one Killing symmetry was also shown. The metric we found is

$$ds_{TST}^2 = \frac{\rho^2}{1 - C^2\rho^4} (d\phi - 4Cz d\tau)^2 + (1 - C^2\rho^4) (d\tau^2 + dz^2 + d\rho^2), \quad (47)$$

where C is a constant parameterizing the non-trivial $SL(2, \mathbb{R})$ group element of the Geroch transformation. The original flat background admits $N = 4$ extended world-sheet supersymmetry. It can be readily checked that this is not the case for the background corresponding to (47) which does not have ordinary extended supersymmetry at all. Nevertheless, a straightforward computation reveals that it has a Yano supersymmetry with

$$F_{TST} = C(1 - C^2\rho^4)\rho^2 d\tau \wedge dz - 4C\rho z d\tau \wedge d\rho + \rho d\phi \wedge d\rho. \quad (48)$$

The same behavior is expected in more general backgrounds in the vicinity of fixed points where the metric can be locally approximated by the flat one written in polar coordinates. The above example

suggests that non-standard, Yano-type supersymmetries might be important when discussing S -duality in a string context.

4 Conclusions

We have discussed the conditions necessary for the existence of approximate symmetries in 2-dim σ -models between collections of string modes. We considered in detail extended supersymmetry among the particle-like modes of the string center of mass as well as among collections of winding and particle-like modes, where we found a new type of supersymmetry. These supersymmetries, though completely different from a field theoretical point of view, are naturally unified under the action of T -duality. Generalization of our discussion by including gauge or other background fields seems conceptually straightforward.

It would be interesting to explore further the behavior of Yano-type supersymmetry under S - or T - S - T -duality and investigate the possibility that it appears as a remnant of ordinary extended supersymmetry after the latter is being destroyed in a conventional sense. This could be relevant in confirming whether or not S -duality is really a symmetry in string theory.

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A Extra susy for the 2d model

In this appendix we provide some of the technical details relevant to the derivation of the conditions for the existence of an extended supersymmetry that were omitted in the main text.

Conservation of the extra supercharge (10) implies a vanishing Poisson bracket with the Hamiltonian (2). A straightforward but tedious computation gives

$$\begin{aligned}
0 &= \{Q_F^-, H\} \\
&= 2D_{[\mu}^- F_{\lambda]\alpha}^- \Pi^\mu X'^\lambda \Psi_-^\alpha - D_\alpha^- F_{\beta\nu}^- \Psi_-^\nu (\Pi^\alpha \Pi^\beta - X'^\alpha X'^\beta) \\
&\quad + \frac{i}{2} (F^-)^\alpha{}_\beta \Psi_-^\beta X'^\lambda (R_{\alpha\lambda\mu\rho}^+ \Psi_+^\mu \Psi_+^\rho - R_{\alpha\lambda\mu\rho}^- \Psi_-^\mu \Psi_-^\rho)
\end{aligned}$$

$$\begin{aligned}
& +\frac{i}{2}(X'^\alpha - \Pi^\alpha)(F^-)^\rho{}_\alpha R_{\mu\nu\lambda\rho}^- \Psi_+^\mu \Psi_+^\nu \Psi_-^\lambda \\
& -\frac{i}{2}(F^-)^\alpha{}_\beta \Psi_-^\beta \Pi^\lambda \left(R_{\alpha\lambda\mu\rho}^+ \Psi_+^\mu \Psi_+^\rho + R_{\alpha\lambda\mu\rho}^- \Psi_-^\mu \Psi_-^\rho \right) \\
& -\frac{1}{4} \left(D_\alpha^+ R_{\mu\nu\rho\lambda}^- + 2H^\sigma{}_{\alpha\rho} R_{\mu\nu\sigma\lambda}^- \right) (F^-)^\alpha{}_\beta \Psi_-^\beta \Psi_-^\rho \Psi_-^\lambda \Psi_+^\mu \Psi_+^\nu \\
& +\frac{1}{4} C_{\mu\nu\rho}^- R_{\alpha\beta\gamma\delta}^+ G^{\mu\alpha} \Psi_-^\nu \Psi_-^\rho \Psi_-^\beta \Psi_-^\gamma \Psi_+^\delta \\
& -\frac{i}{6} \left(\Pi^\alpha + X'^\alpha \right) D_\alpha^- C_{\mu\nu\rho}^- \Psi_-^\mu \Psi_-^\nu \Psi_-^\rho, \tag{49}
\end{aligned}$$

where $\Pi^\mu \equiv \dot{X}^\mu$. Independence of the new supersymmetry from the original $N = 1$ implies the stronger conditions $\{Q_F^-, Q^\pm\} = 0$. Notice that the supersymmetry algebra (8) implies conservation of Q_F^- from these stronger conditions. The Poisson brackets of the candidate extra supercharge Q_F^- with the two standard supercharges are slightly easier to compute. One obtains

$$\begin{aligned}
\{Q_F^-, Q^+\} &= (X'^\alpha - \Pi^\alpha) D_\mu^- F_{\alpha\beta}^- \Psi_+^\mu \Psi_-^\beta \\
& -\frac{i}{2} \left(R_{\alpha\mu\rho\tau}^+ \Psi_+^\rho \Psi_+^\tau + R_{\alpha\mu\rho\tau}^- \Psi_-^\rho \Psi_-^\tau \right) \Psi_+^\mu (F^-)^\alpha{}_\beta \Psi_-^\beta \\
& +\frac{i}{6} D_\mu^- C_{\alpha\beta\gamma}^- \Psi_-^\alpha \Psi_-^\beta \Psi_-^\gamma \Psi_+^\mu \\
& -\frac{i}{6} D_\alpha^+ H_{\mu\nu\rho} \Psi_+^\mu \Psi_+^\nu \Psi_+^\rho (F^-)^\alpha{}_\beta \Psi_-^\beta
\end{aligned} \tag{50}$$

and

$$\begin{aligned}
\{Q_F^-, Q^-\} &= -i \left(\Pi^\alpha \Pi^\beta + X'^\alpha X'^\beta - X'^\alpha \Pi^\beta - \Pi^\alpha X'^\beta \right) F_{\alpha\beta}^- + (F_{\alpha\beta}^- + F_{\beta\alpha}^-) \Psi_-^\alpha \Psi_-^\beta \\
& +\frac{i}{2} \left(R_{\mu\alpha\rho\tau}^+ \Psi_+^\rho \Psi_+^\tau + R_{\mu\alpha\rho\tau}^- \Psi_-^\rho \Psi_-^\tau \right) \Psi_-^\mu (F^-)^\alpha{}_\beta \Psi_-^\beta \\
& +\Pi^\alpha \Psi_-^\mu \Psi_-^\beta \left[D_\beta^- F_{\alpha\mu}^- - H^\rho{}_{\beta\alpha} F_{\rho\mu}^- + \frac{1}{2} F_{\alpha\rho}^- H^\rho{}_{\mu\beta} + \frac{1}{2} C_{\alpha\mu\beta}^- \right] \\
& -X'^\lambda \Psi_-^\mu \Psi_-^\beta \left[D_\beta^- F_{\lambda\mu}^- + D_\lambda^- F_{\mu\beta}^- + (\Omega^-)^\alpha{}_{\lambda\beta} F_{\mu\alpha}^- + (\Omega^-)^\alpha{}_{\beta\lambda} F_{\alpha\mu}^- + \frac{1}{2} F_{\lambda\alpha}^- H^\alpha{}_{\mu\beta} + \frac{1}{2} C_{\lambda\mu\beta}^- \right] \\
& + \left[\frac{i}{6} (F^-)^\rho{}_\mu D_\rho^- H_{\alpha\beta\gamma} + \frac{i}{6} D_\mu^- C_{\alpha\beta\gamma}^- + \frac{i}{4} C_{\rho\alpha\beta}^- H^\rho{}_{\gamma\mu} \right] \Psi_-^\alpha \Psi_-^\beta \Psi_-^\gamma \Psi_-^\mu.
\end{aligned} \tag{51}$$

Notice that this last bracket provides a definition of $C_{\mu\nu\rho}^-$ in terms of $F_{\mu\nu}^-$ when $\{Q_F^-, Q^-\} = 0$ is imposed.

Let us briefly show how to recover the conditions for extended supersymmetries from these expressions. The vanishing of the first line in (49), linear in the fermions, leads to the condition (12) that the tensor $F_{\mu\nu}^-$ is covariantly constant. The vanishing of the first line in (51) forces $F_{\mu\nu}^-$ to be *antisymmetric*, both for the full 2d σ -model, as well as for the limiting cases discussed below. Finally, the vanishing of the terms quadratic in the fermions in (51) provides the relation between the antisymmetric $C_{\mu\nu\rho}^-$ -tensor and $F_{\mu\nu}^-$, given in (14). The vanishing of terms with more than two fermions follows from the integrability of the previously mentioned conditions and the Bianchi identity $D_{[\mu}^\pm R_{\nu\rho]\lambda\alpha}^\pm = \pm H_{[\mu\nu}^\beta R_{\rho]\beta\lambda\alpha}^\pm$.

In the particle-limit (PL) defined by (15), the brackets reduce to

$$\begin{aligned}\{Q_F^-, H\}|_{PL} &= -D_\alpha^- F_{\beta\nu}^- \Psi_-^\nu \Pi^\alpha \Pi^\beta - \frac{i}{2} (F^-)^\alpha{}_\beta \Psi_-^\beta \Pi^\lambda R_{\alpha\lambda\mu\rho}^- \Psi_-^\mu \Psi_-^\rho - \frac{i}{6} \Pi^\lambda D_\lambda^- C_{\mu\nu\rho}^- \Psi_-^\mu \Psi_-^\nu \Psi_-^\rho, \\ \{Q_F^-, Q^-\}|_{PL} &= \Pi^\alpha \Psi_-^\mu \Psi_-^\beta \left[D_\beta^- F_{\alpha\mu}^- - H^\rho{}_{\beta\alpha} F_{\rho\mu}^- + \frac{1}{2} F_{\alpha\rho}^- H^\rho{}_{\mu\beta} + \frac{1}{2} C_{\alpha\mu\beta}^- \right] + \text{4-fermion term},\end{aligned}\quad (52)$$

where the 4-fermion term is made up by the relevant terms in the second and fifth lines in (51). From these, one derives conditions (16) and (17) immediately. Footnote 5 can be read off from the 3-fermion terms in the first equation in (52). Explicit computation using (17) and successive application of the Yano equation leads to

$$\begin{aligned}D_\mu^- C_{\alpha\beta\gamma}^- &= 3R_{\mu\rho[\alpha\beta}^- (F^-)^\rho{}_\gamma + 2D_\sigma^- F_{\alpha[\gamma}^- H^\sigma{}_{\mu]\beta} - 2D_\sigma^- F_{\beta[\gamma}^- H^\sigma{}_{\mu]\alpha} \\ &\quad + \left(3R_{[\alpha\beta\mu]}^-{}^\sigma F_{\sigma\gamma} - (\gamma \leftrightarrow \mu) \right) + \left(3R_{[\mu\gamma\alpha]}^-{}^\sigma F_{\sigma\beta} - (\alpha \leftrightarrow \beta) \right).\end{aligned}\quad (53)$$

The extra terms (compared to footnote 5) can be shown to be proportional to (19), and they vanish by virtue of the condition (18) which is obtained from the 4-fermion terms in (52). In the various manipulations the use of identities such as $D_\mu^- H_{\nu\lambda\rho} = 3R_{\mu[\nu\lambda\rho]}^-$ and $D_{[\mu}^- H_{\nu\rho]\lambda} = R_{[\mu\nu\rho]\lambda}^- - H_{\alpha\lambda[\mu} H_{\nu\rho]}^\alpha$ is necessary.

Similarly the expressions in the winding-particle limit (WPL) defined by (22) are obtained by using $P_0 = 0$ to write

$$\Pi^\alpha|_{WPL} = G^{\alpha\beta} \Pi_\beta|_{WPL} = G^{\alpha i} P_i + G^{\alpha i} B_{i0} X'^0 - \frac{i}{2} G^{\alpha\mu} (\Omega^-)_{\beta;\mu\rho} \Psi_-^\beta \Psi_-^\rho. \quad (54)$$

Inserting this in the brackets leads to the conditions (23), from respectively the $P_i P_j$ term, the $P_i X'^0$ term and the $X'^0 X'^0$ term in the $\{Q_F^-, H\}|_{WPL}$ bracket. The bracket $\{Q_F^-, Q^-\}|_{WPL}$ gives again the expression for $C_{\mu\nu\rho}^-$, as shown in (24).

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